JOURNAL OF AIRCRAFT
Vol. 32, No. 3, May–June 1995

Free Vibration of Composite Beams—an Exact Method Using Symbolic Computation

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An exact dynamic stiffness matrix method has been developed to predict the free vibration characteristics of composite beams (or simple structures assembled from them) for which the bending and torsional displacements are (materially) coupled. To achieve this, an explicit expression is presented for each of the elements of the dynamic stiffness matrix of a bending-torsion coupled composite beam. This was made possible by performing symbolic computing with the help of the package Reduce. Programming the stiffness expressions in Fortran on a SUN SPARC station indicates about 75% savings in computer time when compared with the matrix inversion method normally adopted in the absence of such expressions. The derived dynamic stiffness matrix is then used in conjunction with the Wittrick–Williams algorithm to compute the natural frequencies and mode shapes of composite beams with substantial coupling between bending and torsional displacements. The results obtained from the present theory are compared with those available in the literature and discussed.

Introduction

THIN-WALLED composite beams have aroused continuing research interest because of their substantial benefits and their promise for future application in the aerospace industry. A literature survey revealed a wealth of literature both on the static and the dynamic behavior of such beams with solid as well as thin-walled cross sections. References 1–30 form, in chronological order, a small, carefully selected, sample of this literature, which includes two[16,19] survey papers.

Using linear small deflection theory, this article presents an exact analytical method, called the dynamic stiffness matrix method, to determine the free or forced vibration characteristics of composite beams or of simple structures assembled from them, e.g., a nonuniform composite wing. Of particular interest is the inclusion of the bending-torsion (material) coupling term, which is prevalent in aircraft wings or helicopter blades. Such coupling is common to both solid and thin-walled section composite beams. Recent investigations (which partly motivated the present work), are reported in two excellent papers, one by Minguet and Dugundji[18] and the other by Hodges et al.[27] The authors of both these papers have carried out their research in two parts. In the first part, they established the static stiffness (rigidity) properties of composite beams, and then in the second part, they used them when investigating the free vibration characteristics. Minguet and Dugundji[17,18] for their first part, studied the static behavior both analytically and experimentally. Their analytical model was based on the use of Euler angles and they presented an iterative finite difference procedure to obtain the stiffness properties. Their experimental stiffnesses agreed quite well with their theoretical predictions. Their second part studied the free vibration behavior both theoretically and experimentally. In their theory, they used a standard eigenvalue solution technique based on the influence coefficient method. They included the presence of a static deflection at the tip when obtaining the results for a cantilever. Their experiments were conducted both with and without the inclusion of the static tip deflection and their experimental data compared very well with the results they obtained from their analysis. On the other hand, Hodges et al.[27] obtained the static stiffness properties of composite beams using two qualitatively different methods, of which one is a closed-form analytical method and the other is a detailed cross-sectional finite element method. They also solved the free vibration problem in two different ways, which were based on the mixed formulation of Hodges,[31] one of which is essentially an exact numerical integration method and the other is a mixed finite element method.[27] They presented numerical results on both stiffnesses and natural frequencies, highlighting the differences in results between different methods.

This article, however, uses the dynamic stiffness matrix method to analyze the free vibration characteristics of composite beams or of simple structures composed of them. Some advantages of this method are well known,[32] particularly when higher frequencies and better accuracies are required. An advantage, which is often overlooked but is perhaps more important, is that the method forms a useful comparator when finite element or other approximate methods are used. It should be noted that the theory developed in this article relies on the fact that the rigidity parameters that characterize the composite beam, based on its section properties, are either known or can be found experimentally. Evaluation of these parameters is relatively easier for solid cross section beams, i.e., for flat (plate) beams,[18,27] but is more difficult for thin-walled beams, particularly if they are of open cross section.[23,30] Although the theory presented here is fairly general, it is expected to be more accurate for solid section composite beams than for the thin-walled profile section ones. This is because the effects of additional coupling terms and the effects of shear deformation, rotatory inertia, and warping stiffness terms, which have been neglected in the present theory, are in general much more pronounced for thin-walled composite beams than for their solid section counterparts. However, in appropriate circumstances, the theory can be used to give acceptable

Received Oct. 3, 1993; revision received Oct. 15, 1994; accepted for publication Nov. 15, 1994.

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accuracy for thin-walled cross sections, particularly when they are closed, as well as for solid cross sections. This article appears to be the first to obtain a dynamic stiffness matrix formulation for composite beams.

An explicit dynamic stiffness matrix for a bending-torsion coupled composite beam is derived from the basic governing differential equations. This derivation of explicit stiffness expressions in algebraic form is possible due to the use of symbolic computing.[33,34] The substantial benefit of using the explicit stiffness expressions, as opposed to the numerical computation of the dynamic stiffness matrix by inversion, is illustrated by comparing elapsed CPU times. The application of the derived dynamic stiffness matrix to calculate the natural frequencies and mode shapes of bending-torsion coupled composite beams uses the Wittrick–Williams algorithm.[32,35] Results are given for two illustrative examples for which comparative results are available in the literature.

Theory

Bending-torsion coupled composite beams of both solid and thin-walled cross sections have been characterized in the literature[10,22] by three parameters related to, respectively, the bending rigidity EI, the torsional rigidity GJ, and the bending-torsion material coupling rigidity K, e.g., see Eq. (1) of Ref. 10 or Eq. (1) of Ref. 22. For composite wings[11] or helicopter blades,[8] the bending-torsion material coupling rigidity K is of great significance (whereas it is nonexistent for metallic beams), because it can be exploited to advantage for aeroelastic tailoring.[10,11] There have been several attempts[22,26] to obtain theoretical and experimental values for the rigidities EI, GJ, and K, which are essential to the derivations that follow.

Figure 1 shows a composite beam with a solid rectangular cross section and with a symmetric but unbalanced lay-up. Bending-torsion coupling is well known to occur for such configurations.[6] The beam is assumed to be uniform and straight with length L. In the right-handed axis system shown, the Y axis coincides with the elastic axis, which is permitted bending displacement $h(y, t)$ and torsional rotation $\psi(y, t)$ as indicated, where y is measured from the origin shown and t is time. Using the coupled bending-torsional beam theory for thin-walled composites with shear deformation, rotatory inertia and warping stiffness neglected, the governing differential equations of motion of the beam in free vibration are given by[7]

$$EIh'''' + K\psi''' + m\ddot{h} = 0 \tag{1}$$

$$GJ\psi'' + Kh''' - I_\alpha\ddot{\psi} = 0 \tag{2}$$

where m is the mass per unit length, I_α is the polar mass moment of inertia per unit length about the Y axis, and primes and dots denote differentiation with respect to position y and time t, respectively.

If a sinusoidal variation of h and ψ, with circular frequency ω, is assumed, then

$$\begin{aligned} h(y, t) &= H(y)\sin \omega t \\ \psi(y, t) &= \Psi(y)\sin \omega t \end{aligned} \tag{3}$$

where $H(y)$ and $\Psi(y)$ are the amplitudes of the sinusoidally varying bending displacement and torsional rotation, respectively.

Substituting Eqs. (3) into Eqs. (1) and (2) gives

$$EIH'''' + K\Psi''' - m\omega^2 H = 0 \tag{4}$$

$$GJ\Psi'' + KH''' + I_\alpha\omega^2\Psi = 0 \tag{5}$$

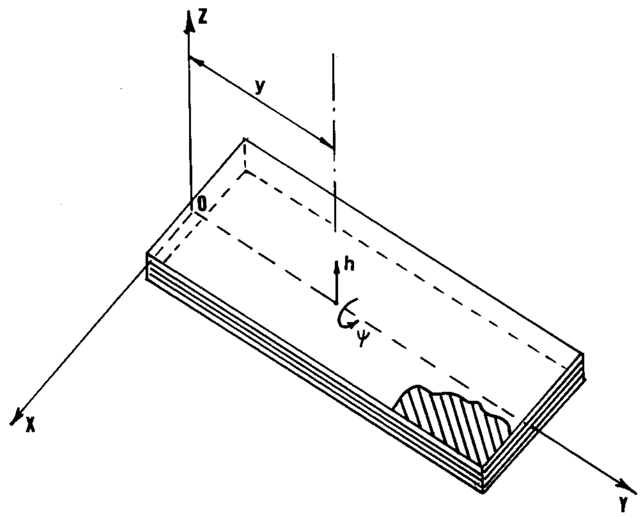

Fig. 1 Coordinate system and notation for a bending-torsion coupled composite beam.

Equations (4) and (5) can be combined into one equation by eliminating either H or Ψ to give

$$(D^6 + aD^4 - bD^2 - abc)W = 0 \tag{6}$$

where

$$W = H \text{ or } \Psi \tag{7}$$

$$D = \frac{\mathrm{d}}{\mathrm{d}\xi}, \qquad \xi = \frac{y}{L} \tag{8}$$

$$\begin{aligned} a &= \bar{a}/c \\ b &= \bar{b}/c \\ c &= 1 - K^2/EIGJ \end{aligned} \tag{9}$$

with

$$\begin{aligned} \bar{a} &= I_\alpha\omega^2L^2/GJ \\ \bar{b} &= m\omega^2L^4/EI \end{aligned} \tag{10}$$

In Eq. (6) a, b, and c are nondimensional quantities and are all positive because it is known that[10,11]

$$0 < c < 1 \tag{11}$$

The solution of the differential Eq. (6) shows that both $H(\xi)$ and $\Psi(\xi)$ have the form

$$\begin{aligned} W(\xi) = {} & C_1 \cosh \alpha\xi + C_2 \sinh \alpha\xi + C_3 \cos \beta\xi \\ & + C_4 \sin \beta\xi + C_5 \cos \gamma\xi + C_6 \sin \gamma\xi \end{aligned} \tag{12}$$

where $W(\xi) = H(\xi)$ or $\Psi(\xi)$, C_1–C_6 are constants, and

$$\begin{aligned} \alpha &= [2(q/3)^{1/2} \cos(\phi/3) - a/3]^{1/2} \\ \beta &= [2(q/3)^{1/2} \cos\{(\pi - \phi)/3\} + a/3]^{1/2} \\ \gamma &= [2(q/3)^{1/2} \cos\{(\pi + \phi)/3\} + a/3]^{1/2} \end{aligned} \tag{13}$$

with

$$\begin{aligned} q &= b + a^2/3 \\ \phi &= \cos^{-1}[(27abc - 9ab - 2a^3)/\{2(a^2 + 3b)^{3/2}\}] \end{aligned} \tag{14}$$

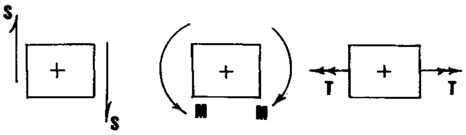

Fig. 2 Sign convention for positive transverse force S, bending moment M, and torque T.

Hence,

$$H(\xi) = A_1 \cosh \alpha\xi + A_2 \sinh \alpha\xi + A_3 \cos \beta\xi + A_4 \sin \beta\xi + A_5 \cos \gamma\xi + A_6 \sin \gamma\xi \tag{15}$$

$$\Psi(\xi) = B_1 \cosh \alpha\xi + B_2 \sinh \alpha\xi + B_3 \cos \beta\xi + B_4 \sin \beta\xi + B_5 \cos \gamma\xi + B_6 \sin \gamma\xi \tag{16}$$

where A_1–A_6 and B_1–B_6 are two different sets of constants.

Substituting Eqs. (15) and (16) into Eq. (4) shows that the constants A_1–A_6 are related to the constants B_1–B_6 by the following relationships:

$$\begin{aligned} B_1 &= (k_\alpha/L)A_2, & B_2 &= (k_\alpha/L)A_1 \\ B_3 &= (k_\beta/L)A_4, & B_4 &= -(k_\beta/L)A_3 \\ B_5 &= (k_\gamma/L)A_6, & B_6 &= -(k_\gamma/L)A_5 \end{aligned} \tag{17}$$

where

$$k_\alpha = (\bar{b} - \alpha^4)/\bar{k}\alpha^3, \qquad k_\beta = (\bar{b} - \beta^4)/\bar{k}\beta^3, \qquad k_\gamma = (\bar{b} - \gamma^4)/\bar{k}\gamma^3 \tag{18}$$

with

$$\bar{k} = K/EI \tag{19}$$

Following the sign convention given in Fig. 2, the anticlockwise rotation $\theta(\xi)$, the bending moment $M(\xi)$, the transverse force $S(\xi)$, and the torque $T(\xi)$ can be obtained from Eqs. (15) and (16) as follows[7] (prime now denotes differentiation with respect to ξ):

$$\theta(\xi) = H'(\xi)/L = (1/L)\{A_1\alpha \sinh \alpha\xi + A_2\alpha \cosh \alpha\xi - A_3\beta \sin \beta\xi + A_4\beta \cos \beta\xi - A_5\gamma \sin \gamma\xi + A_6\gamma \cos \gamma\xi\} \tag{20}$$

$$\begin{aligned} M(\xi) &= -(EI/L^2)H''(\xi) - (K/L)\Psi'(\xi) \\ &= -(EI/L^2)\{H''(\xi) + \bar{k}L\Psi'(\xi)\} \\ &= -(EI/L^2)\{A_1\bar{\alpha} \cosh \alpha\xi + A_2\bar{\alpha} \sinh \alpha\xi - A_3\bar{\beta} \cos \beta\xi - A_4\bar{\beta} \sin \beta\xi - A_5\bar{\gamma} \cos \gamma\xi - A_6\bar{\gamma} \sin \gamma\xi\} \end{aligned} \tag{21}$$

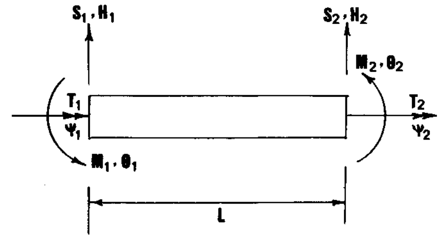

Fig. 3 End conditions for forces and displacements of the beam element.

$$\begin{aligned} S(\xi) &= (EI/L^3)H'''(\xi) + (K/L^2)\Psi''(\xi) \\ &= (EI/L^3)\{H'''(\xi) + \bar{k}L\Psi''(\xi)\} = (EI/L^3)\{A_1\alpha\bar{\alpha} \sinh \alpha\xi + A_2\alpha\bar{\alpha} \cosh \alpha\xi + A_3\beta\bar{\beta} \sin \beta\xi - A_4\beta\bar{\beta} \cos \beta\xi + A_5\gamma\bar{\gamma} \sin \gamma\xi - A_6\gamma\bar{\gamma} \cos \gamma\xi\} \end{aligned} \tag{22}$$

$$\begin{aligned} T(\xi) &= (GJ/L)\Psi'(\xi) + (K/L^2)H''(\xi) = (GJ/L)\{\Psi'(\xi) + (K/GJL)H''(\xi)\} \\ &= (GJ/L^2)\{A_1 g_\alpha \cosh \alpha\xi + A_2 g_\alpha \sinh \alpha\xi - A_3 g_\beta \cos \beta\xi - A_4 g_\beta \sin \beta\xi - A_5 g_\gamma \cos \gamma\xi - A_6 g_\gamma \sin \gamma\xi\} \end{aligned} \tag{23}$$

where

$$\bar{\alpha} = \bar{b}/\alpha^2, \qquad \bar{\beta} = \bar{b}/\beta^2, \qquad \bar{\gamma} = \bar{b}/\gamma^2 \tag{24}$$

$$g_\alpha = (\bar{b} - c\alpha^4)/\bar{k}\alpha^2, \qquad g_\beta = (\bar{b} - c\beta^4)/\bar{k}\beta^2, \qquad g_\gamma = (\bar{b} - c\gamma^4)/\bar{k}\gamma^2 \tag{25}$$

The end conditions for displacements and forces (see Fig. 3) are respectively,

at end 1 (i.e., $\xi = 0$)

$$H = H_1, \qquad \theta = \theta_1 \quad \text{and} \quad \Psi = \Psi_1 \tag{26a}$$

at end 2 (i.e., $\xi = 1$)

$$H = H_2, \qquad \theta = \theta_2 \quad \text{and} \quad \Psi = \Psi_2 \tag{26b}$$

at end 1 (i.e., $\xi = 0$)

$$S = S_1, \qquad M = M_1 \quad \text{and} \quad T = -T_1 \tag{27a}$$

at end 2 (i.e., $\xi = 1$)

$$S = -S_2, \qquad M = -M_2 \quad \text{and} \quad T = T_2 \tag{27b}$$

The dynamic stiffness matrix that relates the amplitudes of the sinusoidally varying forces to the corresponding displacement amplitudes can now be derived with the help of Eqs. (15–27) as follows.

Substituting Eqs. (26) into Eqs. (15), (20), and (16) and using the relationships given by Eqs. (17) gives

$$\begin{bmatrix} H_1 \\ \theta_1 \\ \Psi_1 \\ H_2 \\ \theta_2 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \alpha/L & 0 & \beta/L & 0 & \gamma/L \\ 0 & k_\alpha/L & 0 & k_\beta/L & 0 & k_\gamma/L \\ C_{h\alpha} & S_{h\alpha} & C_\beta & S_\beta & C_\gamma & S_\gamma \\ \alpha S_{h\alpha}/L & \alpha C_{h\alpha}/L & -\beta S_\beta/L & \beta C_\beta/L & -\gamma S_\gamma/L & \gamma C_\gamma/L \\ k_\alpha S_{h\alpha}/L & k_\alpha C_{h\alpha}/L & -k_\beta S_\beta/L & k_\beta C_\beta/L & -k_\gamma S_\gamma/L & k_\gamma C_\gamma/L \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix} \tag{28}$$

i.e.,

$$U = BA \tag{29}$$

where

$$\begin{aligned} C_{h\alpha} &= \cosh \alpha, \quad C_\beta = \cos \beta, \quad C_\gamma = \cos \gamma \\ S_{h\alpha} &= \sinh \alpha, \quad S_\beta = \sin \beta, \quad S_\gamma = \sin \gamma \end{aligned} \tag{30}$$

Substituting Eqs. (27) into Eqs. (22), (21), and (23) gives

$$\begin{bmatrix} S_1 \\ M_1 \\ T_1 \\ S_2 \\ M_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} 0 & W_3\alpha\bar{\alpha} & 0 & -W_3\beta\bar{\beta} & 0 & -W_3\gamma\bar{\gamma} \\ -W_2\bar{\alpha} & 0 & W_2\bar{\beta} & 0 & W_2\bar{\gamma} & 0 \\ -W_1 g_\alpha/L & 0 & W_1 g_\beta/L & 0 & W_1 g_\gamma/L & 0 \\ -W_3\alpha\bar{\alpha}S_{h\alpha} & -W_3\alpha\bar{\alpha}C_{h\alpha} & -W_3\beta\bar{\beta}S_\beta & W_3\beta\bar{\beta}C_\beta & -W_3\gamma\bar{\gamma}S_\gamma & W_3\gamma\bar{\gamma}C_\gamma \\ W_2\bar{\alpha}C_{h\alpha} & W_2\bar{\alpha}S_{h\alpha} & -W_2\bar{\beta}C_\beta & -W_2\bar{\beta}S_\beta & -W_2\bar{\gamma}C_\gamma & -W_2\bar{\gamma}S_\gamma \\ W_1 g_\alpha C_{h\alpha}/L & W_1 g_\alpha S_{h\alpha}/L & -W_1 g_\beta C_\beta/L & -W_1 g_\beta S_\beta/L & -W_1 g_\gamma C_\gamma/L & -W_1 g_\gamma S_\gamma/L \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix} \tag{31}$$

i.e.,

$$F = DA \tag{32}$$

where

$$W_1 = GJ/L, \quad W_2 = EI/L^2, \quad W_3 = EI/L^3 \tag{33}$$

Equations (32) and (29) give

$$F = KU \tag{34}$$

$$\begin{bmatrix} S_1 \\ M_1 \\ T_1 \\ S_2 \\ M_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} & K_{1,5} & K_{1,6} \\ & K_{2,2} & K_{2,3} & K_{2,4} & K_{2,5} & K_{2,6} \\ & & K_{3,3} & K_{3,4} & K_{3,5} & K_{3,6} \\ & & & K_{4,4} & K_{4,5} & K_{4,6} \\ & \text{SYMMETRIC} & & & K_{5,5} & K_{5,6} \\ & & & & & K_{6,6} \end{bmatrix} \begin{bmatrix} H_1 \\ \theta_1 \\ \Psi_1 \\ H_2 \\ \theta_2 \\ \Psi_2 \end{bmatrix} \tag{35}$$

where

$$K = DB^{-1} \tag{36}$$

is the required stiffness matrix.

Equation (36) was solved algebraically with the help of the symbolic computing package Reduce,[33,34] i.e., the B matrix of Eqs. (28) and (29) was inverted algebraically and then premultiplied by the D matrix of Eqs. (31) and (32), again algebraically. The algebraic expressions for each of the 12 independent stiffness elements of the matrix K of Eqs. (35) and (36) were then simplified very considerably, by rigorous further use of symbolic computing,[33,34] to obtain Eqs. (37–71). These stiffness expressions are particularly useful when some, but not all, of them are needed. Thus, the 12 independent stiffness elements of K are presented in concise form in Eqs. (58) if appropriate substitutions are made from Eqs. (37–57) and (59–71), as follows.

Let the following variables be introduced:

$$\mu_1 = \beta k_\alpha - \alpha k_\beta, \quad \mu_2 = \gamma k_\beta - \beta k_\gamma, \quad \mu_3 = \alpha k_\gamma - \gamma k_\alpha \tag{37}$$

$$\bar{\mu}_1 = k_\alpha^2 + k_\beta^2, \quad \bar{\mu}_2 = k_\beta^2 + k_\gamma^2, \quad \bar{\mu}_3 = k_\gamma^2 + k_\alpha^2 \tag{38}$$

$$\nu_1 = \bar{\alpha} + \bar{\beta}, \quad \nu_2 = \bar{\beta} + \bar{\gamma}, \quad \nu_3 = \bar{\gamma} + \bar{\alpha} \tag{39}$$

$$\bar{\nu}_1 = \bar{\alpha} - \bar{\beta}, \quad \bar{\nu}_2 = \bar{\beta} - \bar{\gamma}, \quad \bar{\nu}_3 = \bar{\gamma} - \bar{\alpha} \tag{40}$$

$$\lambda_1 = g_\alpha + g_\beta, \quad \lambda_2 = g_\beta + g_\gamma, \quad \lambda_3 = g_\gamma + g_\alpha \tag{41}$$

$$\bar{\lambda}_1 = g_\alpha - g_\beta, \quad \bar{\lambda}_2 = g_\beta - g_\gamma, \quad \bar{\lambda}_3 = g_\gamma - g_\alpha \tag{42}$$

$$\begin{aligned} \zeta_1 &= k_\alpha + k_\beta - k_\gamma, \quad \zeta_2 = \nu_1 + \nu_3 \\ \zeta_3 &= \lambda_1 + \lambda_3 \end{aligned} \tag{43}$$

$$\begin{aligned} \bar{\zeta}_1 &= \alpha\mu_2\nu_1 + \gamma\mu_1\nu_1, \quad \bar{\zeta}_2 = \bar{\zeta}_1 - \gamma\mu_1\nu_3 \\ \bar{\zeta}_3 &= \alpha\mu_1\bar{\nu}_2 + \gamma\mu_2\nu_1 \end{aligned} \tag{44}$$

$$\begin{aligned} \chi_1 &= \mu_3(\beta - \gamma)k_\gamma + \beta\mu_1 k_\alpha - \gamma\mu_2 k_\beta \\ \chi_2 &= k_\alpha(\beta k_\beta + \gamma k_\gamma), \quad \chi_3 = \lambda_3 - \bar{\lambda}_2 \end{aligned} \tag{45}$$

$$\begin{aligned} \bar{\chi}_1 &= \alpha\bar{\lambda}_2\mu_1 + \gamma\lambda_1\mu_2, \quad \bar{\chi}_2 = \alpha\lambda_1\mu_2 + \gamma\mu_1\zeta_3 \\ \bar{\chi}_3 &= \mu_3\bar{\nu}_2 k_\gamma + \nu_3(\alpha\bar{\mu}_2 - \chi_2) \end{aligned} \tag{46}$$

$$\eta_1 = \nu_3 - \bar{\nu}_2, \quad \eta_2 = \bar{\mu}_2(\alpha^2 + \gamma^2), \quad \eta_3 = \bar{\mu}_3(\beta^2 + \gamma^2) \tag{47}$$

$$\begin{aligned} \bar{\eta}_1 &= \bar{\zeta}_1 + \gamma\mu_1\nu_3, \quad \bar{\eta}_2 = \alpha\eta_1\mu_2 + \gamma\mu_1\zeta_2 \\ \bar{\eta}_3 &= \alpha\mu_2\chi_3 + \gamma\mu_1\zeta_3 \end{aligned} \tag{48}$$

$$\begin{aligned} \sigma_1 &= \bar{\zeta}_2 - 2\gamma\mu_1\bar{\nu}_2, \quad \sigma_2 = \bar{\eta}_2 - 2\gamma\mu_1\nu_1 \\ \sigma_3 &= \alpha\mu_3\bar{\nu}_2 - \beta\mu_2\nu_3 \end{aligned} \tag{49}$$

$$\begin{aligned} \bar{\sigma}_1 &= \alpha\bar{\lambda}_2\mu_3 - \beta\lambda_3\mu_2, \quad \bar{\sigma}_2 = \alpha\lambda_1\mu_2 - \gamma\bar{\lambda}_2\mu_1 \\ \bar{\sigma}_3 &= \alpha\mu_2\chi_3 - \gamma\bar{\lambda}_2\mu_1 \end{aligned} \tag{50}$$

$$\begin{aligned} \rho_1 &= \beta\mu_1\nu_3 - \gamma\mu_3\nu_1, \quad \rho_2 = \beta\lambda_3\mu_1 - \gamma\lambda_1\mu_3 \\ \rho_3 &= \beta\nu_1 k_\alpha - \mu_1\bar{\nu}_1 \end{aligned} \tag{51}$$

$$\begin{aligned} \bar{\rho}_1 &= \mu_1\bar{\nu}_2 k_\alpha + \mu_2\nu_1 k_\gamma, \quad \bar{\rho}_2 = \mu_2\nu_1 k_\alpha + \mu_1\zeta_2 k_\gamma \\ \bar{\rho}_3 &= \mu_2\eta_1 k_\alpha + \mu_1\zeta_2 k_\gamma \end{aligned} \tag{52}$$

$$\begin{aligned} \varepsilon_1 &= \mu_3\bar{\nu}_2 k_\alpha - \mu_2\nu_3 k_\beta, \quad \varepsilon_2 = \bar{\rho}_2 - 2\mu_1\nu_1 k_\gamma \\ \varepsilon_3 &= \bar{\rho}_3 - 2\mu_1\nu_1 k_\gamma \end{aligned} \tag{53}$$

$$\bar{\varepsilon}_1 = \mu_3 k_\alpha + \mu_2 k_\beta, \qquad \bar{\varepsilon}_2 = -\bar{\rho}_1 - \mu_1 \bar{\nu}_3 k_\alpha \qquad \bar{\varepsilon}_3 = \bar{\zeta}_1 + \gamma\mu_1\nu_3 \tag{54}$$

$$\kappa_1 = \gamma\varepsilon_3 + \mu_2\mu_3\bar{\nu}_1 \qquad \kappa_2 = \eta_2 - \eta_3 + 2(\chi_1 - \gamma\mu_3\zeta_1 + \mu_2\mu_3) \qquad \kappa_3 = 2\bar{\alpha}\mu_1\mu_3 + k_\alpha\bar{\varepsilon}_3 \tag{55}$$

$$\bar{\kappa}_1 = \gamma\mu_1\bar{\nu}_2 k_\beta - \mu_2\rho_3 \qquad \bar{\kappa}_2 = \gamma\nu_3\kappa_2 + \gamma\nu_3(\mu_2^2 - \mu_3^2) + \beta\mu_1\mu_3\nu_1 \qquad \bar{\kappa}_3 = \bar{\alpha}(\eta_2 - \eta_3) + \gamma\bar{\varepsilon}_1\bar{\nu}_3 + \beta\bar{\varepsilon}_2 \tag{56}$$

$$\tau_1 = \alpha\beta\mu_1\nu_1 - \alpha\gamma\mu_3\nu_3 + \beta\gamma\mu_2\bar{\nu}_2 \qquad \tau_2 = -(\gamma\mu_1\nu_3 + \beta\mu_3\nu_1)k_\beta \tag{57}$$

where α, β, and γ; $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma}$; k_α, k_β, and k_γ; and g_α, g_β, and g_γ are, respectively, given by Eqs. (13), (24), (18), and (25).

Then

$$\begin{aligned}
K_{1,1} &= K_{4,4} = (EI/L^3)(\Phi_1/\Delta)\\
K_{1,2} &= -K_{4,5} = (EI/L^2)(\Phi_2/\Delta)\\
K_{1,3} &= -K_{4,6} = (EI/L^2)(\Phi_3/\Delta)\\
K_{1,4} &= (EI/L^3)(\Phi_4/\Delta)\\
K_{1,5} &= -K_{2,4} = (EI/L^2)(\Phi_5/\Delta)\\
K_{1,6} &= -K_{3,4} = (EI/L^2)(\Phi_6/\Delta)\\
K_{2,2} &= K_{5,5} = (EI/L)(\Phi_7/\Delta)\\
K_{2,3} &= K_{5,6} = (EI/L)(\Phi_8/\Delta)\\
K_{2,5} &= (EI/L)(\Phi_9/\Delta)\\
K_{2,6} &= K_{3,5} = (EI/L)(\Phi_{10}/\Delta)\\
K_{3,3} &= K_{6,6} = (GJ/L)(\Phi_{11}/\Delta)\\
K_{3,6} &= (GJ/L)(\Phi_{12}/\Delta)
\end{aligned} \tag{58}$$

where

$$\Phi_1 = \mu_2\bar{\zeta}_2 S_\beta S_\gamma C_{h\alpha} + \bar{\kappa}_2 S_\beta C_\gamma S_{h\alpha} - \mu_3\bar{\zeta}_2 C_\beta S_\gamma S_{h\alpha} \tag{59}$$

$$\Phi_2 = \bar{\kappa}_3 S_\beta S_\gamma S_{h\alpha} + \kappa_3 S_{h\alpha}(1 - C_\beta C_\gamma) - \bar{\kappa}_1 S_\beta(1 - C_\gamma C_{h\alpha}) - \kappa_1 S_\gamma(1 - C_\beta C_{h\alpha}) \tag{60}$$

$$\Phi_3 = \tau_1 S_\beta S_\gamma S_{h\alpha} - \alpha\bar{\eta}_1 S_{h\alpha}(1 - C_\beta C_\gamma) - \beta\sigma_1 S_\beta(1 - C_\gamma C_{h\alpha}) + \gamma\sigma_2 S_\gamma(1 - C_\beta C_{h\alpha}) \tag{61}$$

$$\Phi_4 = \mu_3\bar{\zeta}_2 S_\gamma S_{h\alpha} - \bar{\kappa}_2 S_\beta S_{h\alpha} - \mu_2\bar{\zeta}_2 S_\beta S_\gamma \tag{62}$$

$$\Phi_5 = k_\alpha\bar{\zeta}_2 S_{h\alpha}(C_\beta - C_\gamma) - \tau_2 S_\beta(C_\gamma - C_{h\alpha}) + k_\gamma\bar{\zeta}_2 S_\gamma(C_\beta - C_{h\alpha}) \tag{63}$$

$$\Phi_6 = -\bar{\zeta}_2\{\alpha S_{h\alpha}(C_\beta - C_\gamma) - \beta S_\beta(C_\gamma - C_{h\alpha}) + \gamma S_\gamma(C_\beta - C_{h\alpha})\} \tag{64}$$

$$\Phi_7 = \varepsilon_1 S_\beta C_\gamma S_{h\alpha} + \bar{\rho}_1 C_\beta S_\gamma S_{h\alpha} + \bar{\chi}_3 S_\beta S_\gamma C_{h\alpha} - \bar{\rho}_3 C_\beta C_\gamma C_{h\alpha} - \varepsilon_2 C_\beta + \varepsilon_3 C_\gamma + \bar{\rho}_2 C_{h\alpha} \tag{65}$$

$$\Phi_8 = -\sigma_3 S_\beta C_\gamma S_{h\alpha} - \bar{\zeta}_3 C_\beta S_\gamma S_{h\alpha} + \rho_1 S_\beta S_\gamma C_{h\alpha} + \bar{\eta}_2 C_\beta C_\gamma C_{h\alpha} + \sigma_1 C_\beta - \sigma_2 C_\gamma - \bar{\eta}_1 C_{h\alpha} \tag{66}$$

$$\Phi_9 = -\varepsilon_1 S_\beta S_{h\alpha} - \bar{\rho}_1 S_\gamma S_{h\alpha} - \bar{\chi}_3 S_\beta S_\gamma - \varepsilon_2 C_\gamma C_{h\alpha} + \bar{\rho}_2 C_\beta C_\gamma + \varepsilon_3 C_\beta C_{h\alpha} - \bar{\rho}_3 \tag{67}$$

$$\Phi_{10} = \sigma_3 S_\beta S_{h\alpha} + \bar{\zeta}_3 S_\gamma S_{h\alpha} - \rho_1 S_\beta S_\gamma + \sigma_1 C_\gamma C_{h\alpha} - \bar{\eta}_1 C_\beta C_\gamma - \sigma_2 C_\beta C_{h\alpha} + \bar{\eta}_2 \tag{68}$$

$$\Phi_{11} = -\bar{\sigma}_1 S_\beta C_\gamma S_{h\alpha} - \bar{\chi}_1 C_\beta S_\gamma S_{h\alpha} + \rho_2 S_\beta S_\gamma C_{h\alpha} + \bar{\eta}_3 C_\beta C_\gamma C_{h\alpha} + \bar{\sigma}_2 C_\beta - \bar{\sigma}_3 C_\gamma - \bar{\chi}_2 C_{h\alpha} \tag{69}$$

$$\Phi_{12} = \bar{\sigma}_1 S_\beta S_{h\alpha} + \bar{\chi}_1 S_\gamma S_{h\alpha} - \rho_2 S_\beta S_\gamma + \bar{\sigma}_2 C_\gamma C_{h\alpha} - \bar{\chi}_2 C_\beta C_\gamma - \bar{\sigma}_3 C_\beta C_{h\alpha} + \bar{\eta}_3 \tag{70}$$

$$\Delta = \kappa_2 S_\beta S_\gamma S_{h\alpha} - 2\mu_1\mu_2 S_\beta(1 - C_\gamma C_{h\alpha}) - 2\mu_2\mu_3 S_\gamma(1 - C_\beta C_{h\alpha}) + 2\mu_1\mu_3 S_{h\alpha}(1 - C_\beta C_\gamma) \tag{71}$$

It may be noted that because of the bending-torsion (material) coupling effect, the elements $K_{1,3}$, $K_{1,6}$, $K_{2,3}$, $K_{2,6}$, $K_{3,4}$, $K_{3,5}$, $K_{4,6}$, and $K_{5,6}$ of K are not zero, which contrasts with their zero values for the usual Bernoulli–Euler metallic beam, for which the bending and the torsional motions are uncoupled. (Some further acceleration of computation has been deliberately omitted in the coding used to obtain Table 2. For instance, many of the triple products of trigonometric and hyperbolic functions in Eqs. (59–71) appear several times, and so could be computed once and then used repeatedly, e.g., there are four occurrences of each of the products $S_\beta S_\gamma C_{h\alpha}$, $C_\beta S_\gamma S_{h\alpha}$, and $S_\beta C_\gamma S_{h\alpha}$.)

Application of the Dynamic Stiffness Matrix and Results

The dynamic stiffness matrix derived above can be used to compute coupled bending-torsional natural frequencies and mode shapes of composite beams or of simple structures constructed from them, e.g., a nonuniform composite wing is essentially an assembly of such beams. The natural frequencies are calculated by applying the well-known algorithm of Wittrick and Williams,[35] which has subsequently been explained in the literature quite frequently.[32] Therefore, for brevity, it is not explained again here. It is sufficient to note that its application is very simple once the dynamic stiffness matrix of a structure and information about the clamped–clamped natural frequencies of its constituent members are known. For the results given in this article, the clamped–clamped natural frequencies of an individual member were identified as those frequencies at which the Δ of Eq. (71) is zero.

To validate and confirm the accuracy of the theory presented, numerical results follow for two illustrative examples of bending-torsion coupled composite beams available in the literature. The first is the flat composite beam of four ply carbon-fiber reinforced plastic material of Refs. 17 and 18 with [45 deg/0 deg]$_s$ lay-up, length = 0.56 m, width = 0.03 m, and thickness = 0.00054 m. The rigidities and other properties are given in Ref. 18 (see Table 2 on page 1583) as EI = 0.0143 Nm^2, GJ = 0.0195 Nm^2, K = 0.00632 Nm^2, m = 0.0238 kg/m, and I_α = 1.66 × 10^{-6} kgm. For several frequencies, the stiffnesses computed numerically, by performing the matrix inversion and matrix multiplication steps of Eq. (36), were found to agree to machine accuracy with those given by the explicit expressions of Eqs. (58). Note that if the dynamic stiffness matrix is computed numerically by using the matrix inversion and matrix multiplication steps of Eq. (36), care must be taken to use an appropriate inversion routine to find B^{-1}, because one of the diagonal elements of B is zero. Thus, an inversion routine based on Gauss elimination must not be used, but routines such as NAG[36] routine F01AAF can be used. Representative results in Table 1 give the com-

Table 1 Numerical values of the dynamic stiffness matrix elements of the bending-torsion coupled composite beam of example 1

Stiffness terms	Numerical values, ω = 25 rad/s	Numerical values, ω = 90 rad/s
$K_{1,1}$ and $K_{4,4}$	−2.728020472	−17.10535656
$K_{1,2}$ and $-K_{4,5}$	−0.06567085301	0.01431729924
$K_{1,3}$ and $-K_{4,6}$	−0.0001318398455	−0.004441336143
$K_{1,4}$	−2.342655453	17.27414778
$K_{1,5}$ and $-K_{2,4}$	0.4318486623	−1.536939668
$K_{1,6}$ and $-K_{3,4}$	−0.0002049877195	−0.00004575968013
$K_{2,2}$ and $K_{5,5}$	0.05963650395	0.1424613240
$K_{2,3}$ and $K_{5,6}$	0.01124295252	0.01039354865
$K_{2,5}$	0.06519495214	−0.1415366399
$K_{2,6}$ and $K_{3,5}$	−0.01133726231	−0.01164304991
$K_{3,3}$ and $K_{6,6}$	0.03462754318	0.03227474241
$K_{3,6}$	−0.03491844778	−0.03610896528

Table 2 CPU time on a SUN SPARC station using Fortran

Number of iterations (number of frequencies)	CPU time, s: Explicit expressions	CPU time, s: Numerical (inversion) method
500	0.23	1.19
1000	0.49	2.26
2000	0.95	4.39
5000	2.37	11.47

puted stiffnesses for two frequency values. The 12 independent stiffnesses shown were computed using double precision arithmetic in Fortran and are presented to 10 significant figures so as to form a datum to enable other interested workers to check their own member stiffness expressions or their computer coding of Eqs. (37–71). Programming the explicit stiffness expressions has substantial computational advantage over the numerical method, i.e., the matrix inversion and multiplication steps of Eq. (36). This is evident from the recorded elapsed CPU time on a SUN SPARC station shown in Table 2. The comparison was made for numerous iterations, each performed at a different frequency, and shows that programming the explicit expressions has a more than fourfold advantage over the inversion method.

To complete this first example, the first five natural frequencies and mode shapes of the beam, with cantilever end conditions, were calculated and are shown in Fig. 4. The calculated natural frequencies of Ref. 17 (using its results for the zero static tip deflection case) are shown in parentheses and clearly agree quite well. As expected, the corresponding mode shapes of Fig. 4 also resemble the ones given in Fig. 5.16 of Ref. 17 so closely that they are impossible to draw separately on Fig. 4. Also, to be consistent with Ref. 17, the torsional rotation ψ was multiplied by the chord length $\bar{c}$ (= 30 mm) when plotting the mode shapes. For the first two natural frequencies, the mode shapes of the beam are dominated by bending displacements, whereas for the next two there is more coupling between the bending and the torsional displacements, and the fifth natural frequency gives the fundamental torsional mode, as was previously observed by Migunet and Dugundji.[17,18]

The second illustrative example is to find the first four natural frequencies of the similar bending-torsion coupled cantilever beam independently investigated first by Migunet and Dugundji[18] and then by Hodges et al.[27] This beam is a thin 12-ply strip of carbon fiber reinforced plastic material with [45 deg/0 deg]$_{3s}$ lay-up and of length 0.56 m. The rigidities were calculated[27] using a program based on Ref. 5 and called nonhomogeneous anisotropic beam section analysis (NABSA). These rigidities and other properties of the beam given in Ref. 27 were used after conversion from the earlier[27] Imperial

Table 3 Natural frequencies of the bending-torsion coupled cantilever composite beam of example 2

Frequency no.	Frequency, Hz: Experimental[18]	Frequency, Hz: Theoretical[27]	Frequency, Hz: Present theory
1	4.3	4.66	4.66
2	28	29.60	29.17
3	78	84.89	81.63
4	135	113.43	113.28

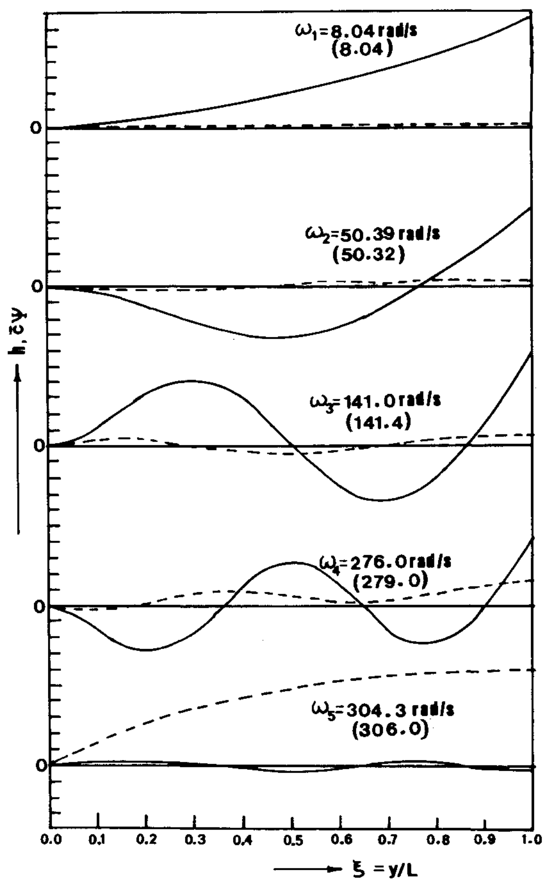

Fig. 4 Coupled bending-torsional natural frequencies (with Ref. 17 results in parentheses) and mode shapes of a composite beam. Key: —— *h*; -------- $\bar{c}\psi$.

units to *SI* ones, which gave: EI = 0.5317 Nm^2, GJ = 0.3586 Nm^2, K = 0.0990 Nm^2, m = 0.07383 kg/m, and I_α = 5.562 × 10^{-6} kgm. Table 3 compares the present results with the experimental natural frequencies of Ref. 18 and the finite element natural frequencies of Ref. 27. Apart from the modest differences with experimental results, the agreement is generally very good. However, caution is advised when using the theory for thin-walled composite beams, for which its use becomes questionable when terms related to additional couplings, shear deformation, rotatory inertia, and warping stiffness are significant. Nevertheless, for many practical composite beams with thin-walled closed cross sections, such as box and aerofoil sections, acceptable accuracy can be achieved using the present theory. In contrast, extra caution is needed when considering thin-walled open cross section composite beams.

Conclusions

A dynamic stiffness matrix has been developed for a composite beam which exhibits material coupling between bending and torsional displacements. Rigorous use of symbolic computing has yielded explicit expressions for the elements of the dynamic stiffness matrix in concise algebraic form. Programming the explicit expressions offers substantial savings in computer time when compared to the alternative numerical method based on matrix inversion and multiplication. Application of the dynamic stiffness matrix in conjunction with the Wittrick–Williams algorithm enables prediction of the natural frequencies and mode shapes of composite beams or of simple structures containing them, when the beams exhibit material coupling between bending and torsional displacements. Two illustrative examples show good agreement with published results. Care should be exercised when applying the theory to composite beams of thin-walled cross section, particularly in the case of open sections.

Acknowledgments

The authors are grateful to John Dugundji and Dewey Hodges for providing useful references on the subject.

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